Notes on Convex Sets

4 July 2010

These are a few definitions and quick observations that may be helpful when reading Vic Klee's "Unsolved Problems in Intuitive Geometry". Everything is taken from the introductory chapters of [1]. We will be working exclusively in the n-dimensional Euclidean space E^n.

Definition 1. *A set C is **convex** if for every pair of points $x, y \in C$, the closed line segment from x to y is contained in C. That is, C contains the point $tx + (1-t)y$ for every $t \in [0,1]$. We call C a **convex body** if it is convex and compact.*

It is immediately clear from the definition that the intersection of any family of convex sets is also convex. As a result, the next definition makes sense.

Definition 2. *The **convex hull** of a set $X \subset E^n$ is the intersection of all the convex sets of E^n containing X. Equivalently, the convex hull of a nonempty set $X \subset E^n$ is the set of **convex combinations***

$$\sum_{i=1}^{k} \alpha_i x_i, \qquad \text{where} \quad x_i \in X,\ \alpha_i \geq 0,\ \sum_{i=1}^{k} \alpha_i = 1,\ k \in [n].$$

Thus we can view the convex hull of a set as the "smallest" convex set containing it. It is important to note that the convex hull of a compact set is a convex body. In particular, the convex hull of a finite set, a **polyhedron**[1], is a convex body.

We now take a look at some of the creatures that interact with our convex sets.

Definition 3. *A set $F \subset E^n$ is a **flat** (or affine variety) of dimension k if it takes the form $x + V$ where V is a subspace of dimension k. A **hyperplane** is an $(n-1)$-dimensional flat.*

Note that the intersection of a polyhedron with a flat is also a polyhedron.

A hyperplane H **cuts** a set X if both open halfspaces determined by H contain points in X. A hyperplane H **supports** X if H does not cut X, but the distance $d(X,H) = \inf\{||x-h|| \mid x \in X, h \in H\} = 0$. If X is a convex set in E^n, then F is a **face** of X if $F = \emptyset$, $F = X$ or there is a supporting hyperplane H of X where $F = H \cap X$.

Definition 4. *A **projective transformation** is a map $T : E^n \to E^n$ where*

$$Tx = \frac{Ax + b}{\langle c, x\rangle + d},$$

*where A is a linear transformation of E^n to itself (taken to be a matrix), b and c are n-dimensional row vectors, d is a scalar, and either c or d is nonzero. A projective transformation is called an **affine transformation** if the denominator $\langle c, x\rangle + d$ is nonzero for all x. It is called **nonsingular** if $\begin{pmatrix} A & b^T \\ c & d \end{pmatrix}$ is invertible.*

If $A, B \subset E^n$, they are said to be **affinely equivalent** if there is a nonsingular affine transformation from A onto B.

The image of a convex set under an affine transformation is also a convex set. Furthermore, the affine image of a polyhedron is also a polyhedron.

[1]Some authors reserve the term "polyhedron" for 3-dimensional bodies, and use the term "polytope" for the general case.

Definition 5. *A set* $\{x_1, \ldots, x_k\}$ *is* ***affinely dependent*** *if there is a linear combination* $\sum_{i=1}^{k} a_i x_i = 0$ *where some* $a_i \neq 0$ *and* $\sum_{i=1}^{k} a_i = 0$*. An* n*-**simplex** is the convex hull of* $n+1$ *affinely independent points.*

It is easy to see that each pair of n-simplices is affinely equivalent.

References

[1] Branko Grünbaum. *Convex Polytopes*, Second Edition. Springer-Verlag, New York, 2003.